Numerical Appendix to "The Declining Equity Premium: What Role Does Macroeconomic Risk Play?"*

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Numerical Appendix to "The Declining Equity Premium: What Role Does Macroeconomic Risk Play?"

1 Numerical Solutions

This appendix describes the algorithm used to solve for prices in the paper "The Declining Equity Premium: What Role Does Macroeconomic Risk Play?"

1.1 Pricing the Consumption and Dividend Claims, Computation of Unconditional Equity Premium: Learning Model

The first order conditions that the price-dividend ratio of a claim to the dividend stream satisfies

$$E_t\left[M_{t+1}\left(\frac{P^D_{t+1}}{D_{t+1}}(\hat{\xi}_{t+2|t+1})+1\right)\frac{D_{t+1}}{D_t}\right]=\frac{P^D_t}{D_t}(\hat{\xi}_{t+1|t}), \tag{1}$$

and the price-consumption ratio for the consumption claim satisfies

$$E_t\left[M_{t+1}\left(\frac{P^C_{t+1}}{C_{t+1}}(\hat{\xi}_{t+2|t+1})+1\right)\frac{C_{t+1}}{C_t}\right]=\frac{P^C_t}{C_t}(\hat{\xi}_{t+1|t}). \tag{2}$$

Notice that $\frac{P^C_t}{C_t}$ is the wealth-consumption ratio, where wealth here is measured on an ex-dividend basis. The posterior probabilities $\hat{\xi}_{t+1|t}$ are the only state variables in this framework, so the price-dividend ratio is a function only of $\hat{\xi}_{t+1|t}$. We solve these functional equations numerically on a grid of values for the state variables $\hat{\xi}_{t+1|t}$.

Equation (1) can be rewritten as:

$$E_t\left[M_{t+1}\left(\frac{P^D_{t+1}}{C^\lambda_{t+1}}+1\right)\left(\frac{C_{t+1}}{C_t}\right)^\lambda\right]=\frac{P^D_t}{C^\lambda_t}. \tag{3}$$

Applying the definition of returns with $\lambda=1$, M_{t+1} can be rewritten as

$$M_{t+1}=\delta^\alpha\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha-1}\left(\frac{P^C_{t+1}}{C_{t+1}}+1\right)^{\alpha-1}\left(\frac{P^C_t}{C_t}\right)^{1-\alpha}. \tag{4}$$

Plugging (4) into (3) we obtain

$$E_t\left[\delta^\alpha\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha-1}\left(\frac{P^C_{t+1}}{C_{t+1}}+1\right)^{\alpha-1}\left(\frac{P^C_t}{C_t}\right)^{1-\alpha}\left(\frac{P^D_{t+1}}{C^\lambda_{t+1}}+1\right)\left(\frac{C_{t+1}}{C_t}\right)^\lambda\right]=\frac{P^D_t}{C^\lambda_t} \tag{5}$$

The price-dividend ratio on equity, $\frac{P^D_t}{C^\lambda_t}$, is defined recursively by (5).

We write the price-dividend ratio for a levered consumption claim as a function of the state vector $\hat{\xi}_{t+1|t}$:

$$\frac{P_t^D}{C_t^\lambda} = F_D(\hat{\xi}_{t+1|t}). \tag{6}$$

Similarly, the price-dividend ratio for the unlevered consumption claim can be written

$$\frac{P_t^C}{C_t} = F_C(\hat{\xi}_{t+1|t}), \tag{7}$$

for some function F_C. Notice that the price-dividend ratio in (7) is simply the wealth-consumption ratio, where wealth is defined to be ex-dividend wealth.

The wealth-consumption ratio is defined as the fixed point of (5) for $\lambda = 1$ and $P^D = P^C$ everywhere. Applying this case to (5) and substituting $\frac{P_t^C}{C_t} = F_C(\hat{\xi}_{t+1|t})$, the Euler equation for the consumption claim may be written

$$E_t\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha\right] = \left(F_C(\hat{\xi}_{t+1|t})\right)^\alpha.$$

From the definition of the conditional expectation, the left-hand-side of the expression above is given by

$$\begin{aligned} &E_t\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha\right] = \\ &\qquad \sum_{j=1}^{N} P\left\{s_{t+1}=j \mid \mathbf{Y}_t\right\} E\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)_j^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha \mid s_{t+1}=j, \mathbf{Y}_t\right], \end{aligned} \tag{8}$$

where $\log\left[\left(\frac{C_{t+1}}{C_t}\right)_j\right] \equiv \mu(s_j)+\sigma(s_j)\epsilon_{t+1}$ denotes consumption growth in state j. The distribution of $\hat{\xi}_{t+2|t+1}$ conditional on time-t data Y_t and on the state s_{t+1} depends only on $\hat{\xi}_{t+1|t}$ and the state. Using the definition of $\hat{\xi}_{t+1|t}$, (8) can be written

$$\begin{aligned} &E_t\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha\right] = \\ &\qquad \sum_{j=1}^{N} \hat{\xi}_{t+1|t}(j) E\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)_j^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha \mid s_{t+1}=j, \hat{\xi}_{t+1|t}\right]. \end{aligned}$$

Thus, the wealth-consumption ratio, $F_C(\hat{\xi}_{t+1|t})$, is defined by the recursion:

$$\left(F_C(\hat{\xi}_{t+1|t})\right)^\alpha = \sum_{j=1}^{N} \hat{\xi}_{t+1|t}(j) E\left[\delta^\alpha \left(\frac{C_{t+1}}{C_t}\right)_j^{-\frac{\alpha}{\psi}+\alpha} \left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^\alpha \mid s_{t+1}=j, \hat{\xi}_{t+1|t}\right]. \tag{9}$$

It is straightforward to show that a similar recursion defines the price-dividend ratio of a levered equity claim, $F_D(\hat{\xi}_{t+1|t})$, by allowing λ to take on arbitrary values greater than unity. From (5), we have

$$\frac{P_t^D}{D_t} = F_C(\hat{\xi}_{t+1|t})^{1-\alpha}\delta^\alpha E_t\left[\left(\frac{C_{t+1}}{C_t}\right)_j^{-\frac{\alpha}{\psi}+\alpha-1+\lambda}\left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^{\alpha-1}\left(\frac{P_{t+1}^D}{D_{t+1}}+1\right)\right].$$

Substituting (6) into the above, we obtain

$$F_D(\hat{\xi}_{t+1|t}) = F_C(\hat{\xi}_{t+1|t})^{1-\alpha}\delta^\alpha\times$$
$$\sum_{j=1}^N \hat{\xi}_{t+1|t}(j)E\left[\left(\frac{C_{t+1}}{C_t}\right)_j^{-\frac{\alpha}{\psi}+\alpha-1+\lambda}\left(F_C(\hat{\xi}_{t+2|t+1})+1\right)^{\alpha-1}\left(F_D(\hat{\xi}_{t+2|t+1})+1\right) \mid s_{t+1}=j, \hat{\xi}_{t+1|t}\right].$$

The expectation above is computed by numerical integration under the assumption that innovations to consumption growth are i.i.d., conditional on the state j.

Denote the log return of the dividend claim from t to $t+1$ as $\log\left(R_{D,t+1}\right) = r_{D,t+1}$. It is also possible to compute moments of log returns:

$$E_t[r_{D,t+1}] = E_t\left[\log\left(\frac{F_D(\hat{\xi}_{t+2|t+1})+1}{F_D(\hat{\xi}_{t+1|t})}\left(\frac{D_{t+1}}{D_t}\right)\right)\right] \tag{10}$$

$$\sigma_t^2[r_{D,t+1}] = E_t\left[\left(\log\left(\frac{F_D(\hat{\xi}_{t+2|t+1})+1}{F_D(\hat{\xi}_{t+1|t})}\left(\frac{D_{t+1}}{D_t}\right)\right)\right)^2\right] - \left(E_t\left[r_{D,t+1}\right]\right)^2$$

Equation (10) points the way to calculating the expected L period return. For example, suppose we were interested in the annualized compound rate of return from investing in the levered consumption claim for thirty years. Because the model is calibrated to t equals a quarter, we would compute

$$\frac{4}{120}E_t[r_{D,t+1}+r_{D,t+2}+\cdots+r_{D,t+120}] \tag{11}$$

To compute the thirty-year equity premium, we could subtract out the return from rolling over investments in the risk-free rate. Let $r_{t+1}^f = \log R_{t+1}^f$, then

$$\frac{4}{120}E_t[r_{t+1}^f+r_{t+2}^f+\cdots+r_{t+120}^f] \tag{12}$$

Note that r_{t+1}^f is known at time t and so could be brought outside the expectation. Subtracting (12) from (11) gives the thirty-year ahead risk premium.

The question is how to compute the elements in the sums of (11) and (12)? We show that these quantities can be computed recursively. Because the one-period ahead expected return and risk-free rate are functions of $\hat{\xi}_{t+1|t}$, we can write

$$G_1(\hat{\xi}_{t+1|t}) = E_t(r_{D,t+1})$$

Define

$$G_2(\hat{\xi}_{t+1|t}) = E_t(G_1(\hat{\xi}_{t+2|t+1}))$$

By the law of iterated expectations

$$G_2(\hat{\xi}_{t+1|t}) = E_t(G_1(\hat{\xi}_{t+2|t+1})) = E_t(E_{t+1}(r_{D,t+2})) = E_t(r_{D,t+2})$$

More generally, define G_m recursively as

$$G_m(\hat{\xi}_{t+1|t}) = E_t(G_{m-1}(\hat{\xi}_{t+2|t+1}))$$

Note that assuming $G_{m-1}(\hat{\xi}_{t+1|t}) = E_t[r_{t+m-1}]$ implies,

$$G_m(\hat{\xi}_{t+1|t}) = E_t(G_{m-1}(\hat{\xi}_{t+2|t+1})) = E_t(E_{t+1}(r_{D,t+m})) = E_t(r_{D,t+m}). \tag{13}$$

So by induction, (13) holds for all L.

Similarly define

$$H_1(\hat{\xi}_{t+1|t}) = r^f_{t+1}$$

and

$$H_m(\hat{\xi}_{t+1|t}) = E_t(H_{m-1}(\hat{\xi}_{t+2|t+1}))$$

If $H_{m-1}(\hat{\xi}_{t+1|t}) = E_t[r^f_{t+m-1}]$,

$$H_m(\hat{\xi}_{t+1|t}) = E_t(H_{m-1}(\hat{\xi}_{t+2|t+1})) = E_t(E_{t+1}(r^f_{t+m})) = E_t(r^f_{t+m}). \tag{14}$$

So (14) holds for all L. Thus the L-period ahead expected return can be found by recursively calculating G_m for $m = 1, \ldots L$, summing up, and multiplying by $4/L$. The L-period ahead risk premium can be found by calculating H_m for $m = 1, \ldots L$, summing up the differences $G_m - H_m$, and multiplying by $4/L$.

1.2 III. Pricing the Consumption and Dividend Claims: Cointegration Model

To solve the cointegration model, rewrite the cointegrated system as

$$\begin{aligned}
\Delta d_{t+1} &= \mu + z_t + \lambda \epsilon_{c,t+1} + \epsilon_{d,t+1} \\
z_{t+1} &= (1-k) z_t + k(1-\lambda)\epsilon_{c,t+1} - k\epsilon_{d,t+1},
\end{aligned}$$

The price-consumption ratio P_t/C_t solves the following:

$$E_t\left[\delta^{\alpha}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha}\left(\frac{P^C_{t+1}}{C_{t+1}}+1\right)^{\alpha}\right]=\left(\frac{P^C_t}{C_t}\right)^{\alpha}$$

This equation is solved by

$$\frac{P^C_t}{C_t}=\frac{R}{1-R}, \tag{15}$$

where

$$R=\delta\exp\left\{\left(1-\frac{1}{\psi}\right)\mu+\frac{1}{2}\alpha\left(1-\frac{1}{\psi}\right)^2\sigma_c^2\right\}.$$

The price-dividend ratio then solves the equation

$$E_t\left[\delta^{\alpha}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\alpha}{\psi}+\alpha-1}R^{1-\alpha}\left(\frac{D_{t+1}}{D_t}\right)\left(\frac{P^D_{t+1}}{D_{t+1}}+1\right)\right] \tag{16}$$

This follows from substituting (15) into the equation for the price-dividend ratio. Conjecture that the price-dividend ratio has a solution of the form

$$\frac{P^D_t}{D_t}=\sum_{n=1}^{\infty}\exp\left\{A_n+B_n z_t\right\}. \tag{17}$$

Substituting (17) into (16) verifies the conjecture and implies that A_n and B_n satisfy

$$\begin{aligned}A_n &= A_{n-1}+\alpha\log\delta+(1-\alpha)\log R+\alpha\left(1-\frac{1}{\psi}\right)\mu+ \\ &\qquad \frac{1}{2}\left(\alpha\left(1-\frac{1}{\psi}\right)+(\lambda-1)(1-B_{n-1}k)\right)^2\sigma_c^2+\frac{1}{2}\left(1-B_{n-1}k\right)^2\sigma_d^2 \qquad (18)\\ B_n &= (1-k)B_{n-1}+1, \qquad (19)\end{aligned}$$

with boundary conditions $A_0=B_0=0$.