Appendix for "Elasticities of Substitution in Real Business Cycle Models with Home Production"*

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Appendix: Approximate Loglinear Solutions for the paper Elasticities of Substitution in Real Business Cycle Models with Home Production By John Y. Campbell and Sydney C. Ludvigson

This appendix provides illustrative solutions to the loglinearized model for Cases 1 through 4. Cases 5 and 6 are straightforward generalizations of Cases 3 and 4. We use the method of Campbell (1994). Here we provide only the solutions for the elasticities, and refer the reader to Campbell (1994) for details about the procedure. In each case, the model's equations are made linear in logs by approximating them with first order Taylor expansions around steady state values. We start with the most general case and proceed backwards. Throughout this appendix we use the notation $\sigma = 1/\gamma$.

Case 4

Combining (??) and (??) we get an accumulation equation for $F_t \equiv K_t + D_t$:

$$F_{t+1} = (1 - \delta)F_t + Y_t - C_t.$$
(A.1)

Taking logs of both sides and linearizing the right hand side yields an equation for f_{t+1} :

$$f_{t+1} = \lambda_1 k_t + \lambda_2 (a_t + n_t) + \lambda_3 c_t \lambda_4 f_t, \qquad (A.2)$$

where,

$$\lambda_1 \equiv \frac{(r+\delta)N}{1+g}, \quad \lambda_2 \equiv \frac{(r+\delta)N\alpha}{(1+g)(1-\alpha)}, \quad \lambda_3 \equiv \frac{(\delta+g)}{1+g} - \frac{(r+\delta)N}{(1+g)(1-\alpha)}, \quad \lambda_4 \equiv 1 - \frac{\delta+g}{1+g}$$

log-linearizing the work-wage first-order condition (??) yields an equation for log hours:

$$n_t = \nu_1 k_t + \nu_2 d_t + \nu_3 a_t + \nu_4 c_t, \tag{A.3}$$

where,

$$\nu_1 \equiv \nu^*(1-\alpha), \quad \nu_2 \equiv \nu^*(1/\sigma - 1)(1-\alpha), \quad \nu_3 \equiv \nu^*\alpha/\sigma, \quad \nu_4 \equiv -\nu^*/\sigma,$$

and where,

$$\nu^* \equiv (\nu(1-N)\sigma)/((1-N)\sigma + \nu\alpha N), \quad \nu \equiv (1-N)/(1-\alpha).$$

Equation (??) is loglinearized assuming that R_{t+1} and C_{t+1} are jointly lognormal and homoskedastic to obtain:

$$E_t \Delta c_t = E_t [\xi_1 k_{t+1} + \xi_2 d_{t+1} + \xi_3 a_{t+1} + \xi_4 c_{t+1}], \qquad (A.4)$$

where,

$$\begin{split} \xi_1 &\equiv (\sigma \alpha (r+\delta)(\nu_1-1))/(1+r), \quad \xi_2 &\equiv (\sigma \alpha (r+\delta)\nu_2)/(1+r) \\ \xi_3 &\equiv (\sigma \alpha (r+\delta)(\nu_3+1))/(1+r), \quad \xi_4 &\equiv (\sigma \alpha (r+\delta)\nu_4)/(1+r). \end{split}$$

We assume that individuals can reallocate capital between the home and market sectors within the period. This allows us to equate the gross marginal products of each type of capital in (??) and (??) yielding an equation for k_t and d_t in terms of f_{t,a_t} , and c_t :

$$k_t = \pi_1 f_t + \pi_2 a_t + \pi_3 c_t \tag{A.5}$$

$$d_t = \chi_1 f_t + \chi_2 a_t + \chi_3 c_t, \tag{A.6}$$

where,

$$\chi_1 \equiv \omega_1^* \pi_1, \quad \chi_2 \equiv \omega_1^* \pi_2 + \omega_2^*, \quad \chi_3 \equiv \omega_1^* \pi_3 + \omega_3^*,$$
$$\omega_1^* \equiv (\omega_1 \nu_1 + \omega_3) / (1 - \omega_1 \nu_2), \quad \omega_2^* \equiv (\omega_1 \nu_3 + \omega_2) / (1 - \omega_1 \nu_2),$$
$$\omega_3^* \equiv (\omega_1 \nu_4 + \omega_4) / (1 - \omega_1 \nu_2),$$

and where,

$$\omega_1 \equiv -(N/(1-N)\alpha(1-1/\sigma)\sigma/((1-\alpha)+\alpha\sigma) - \alpha\sigma/((1-\alpha)+\alpha\sigma),$$

$$\omega_2 \equiv -\alpha/((1-\alpha)+\alpha\sigma), \quad \omega_3 \equiv \alpha\sigma/((1-\alpha)+\alpha\sigma), \quad \omega_4 \equiv 1/((1-\alpha)+\alpha\sigma),$$

$$\pi_1 \equiv (1/N)/(1+(1-N)\omega_1^*/N), \quad \pi_2 \equiv -(1-N)\omega_2^*/(N+(1-N)\omega_1^*),$$

$$\pi_3 \equiv -\omega_3^*(1-N)/(N+(1-N)\omega_1^*).$$

The solution proceeds by the method of undetermined coefficients, by making an initial guess that the loglinear solution will be of the form specified in (??).

 η_{cf} solves the quadratic equation:

$$Q_2\eta_{cf}^2 + Q_1\eta_{cf} + Q_0 = 0, (A.7)$$

where,

$$Q_2 \equiv (\psi_3 \mu_3 - \mu_3), \quad Q_1 \equiv (1 + \psi_1 \mu_3 + \psi_3 \mu_1 - \mu_1), \quad Q_0 \equiv \psi_1 \mu_1.$$

where,

$$\begin{split} \psi_1 &\equiv \xi_1 \pi_1 + \xi_2 \chi_1, \quad \psi_2 &\equiv \xi_1 \pi_2 + \xi_2 \chi_2 + \xi_3, \quad \psi_3 &\equiv \xi_1 \pi_3 + \xi_2 \chi_3 + \xi_4, \\ \text{and where,} \end{split}$$

$$\mu_1 \equiv \lambda_1^* \pi_1 + \lambda_3^* \chi_1 + \lambda_5^*, \quad \mu_2 \equiv \lambda_1^* \pi_2 + \lambda_3^* \chi_2 + \lambda_2^*, \quad \mu_3 \equiv \lambda_1^* \pi_3 + \lambda_3^* \chi_3 + \lambda_4^*,$$

where,

$$\lambda_1^* \equiv \lambda_1 + \lambda_2 \nu_1, \quad \lambda_2^* \equiv \lambda_2 + \lambda_2 \nu_3, \quad \lambda_3^* \equiv \lambda_2 \nu_2, \quad \lambda_4^* \equiv \lambda_3 + \lambda_2 \nu_4, \quad \lambda_5^* \equiv \lambda_4.$$

 η_{ca} is given by

$$\eta_{ca} = \frac{-(\psi_1 \mu_2 + \psi_3 \mu_2 \eta_{cf} - \mu_2 \eta_{cf} + \psi_2 \phi)}{\psi_1 \mu_3 + \psi_3 \mu_3 \eta_{cf} - \mu_3 \eta_{cf} + \psi_3 \phi + 1 - \phi}$$
(A.8)

Elasticities of total capital with respect to last period's total capital and the log technology shock are then found as

$$\eta_{ff} = \mu_1 + \mu_3 \eta_{cf}, \quad \eta_{fa} = \mu_2 + \mu_3 \eta_{ca}.$$

All other elasticities are defined in terms of the elasticities above:

$$\begin{split} \eta_{kf} &= \pi_1 + \pi_3 \eta_{cf}, \quad \eta_{ka} = \pi_2 + \pi_3 \eta_{ca}, \quad \eta_{df} = \chi_1 + \chi_3 \eta_{cf}, \quad \eta_{da} = \chi_2 + \chi_3 \eta_{ca}, \\ \eta_{nf} &= \nu_1 \eta_{kf} + \nu_2 \eta_{df} + \nu_4 \eta_{cf}, \quad \eta_{na} = \nu_1 \eta_{ka} + \nu_2 \eta_{da} + \nu_3 + \nu_4 \eta_{ca}, \\ \eta_{yf} &= \alpha \eta_{nf} + (1 - \alpha) \eta_{kf}, \quad \eta_{ya} = \alpha (1 + \eta_{na}) + (1 - \alpha) \eta_{ka}, \\ \eta_{hf} &= (1 - \alpha) \eta_{df} - \alpha N \eta_{nf} / (1 - N), \quad \eta_{ha} = (1 - \alpha) \eta_{da} + \alpha - \alpha N \eta_{na} / (1 - N). \\ \mathbf{Case \ 3} \end{split}$$

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Parameter definitions for Case 3 are the same as in Case 4, with the following exceptions:

$$\begin{split} \omega_2 &\equiv -\alpha \sigma / ((1-\alpha) + \alpha \sigma), \quad \nu_3 \equiv \nu^* \alpha, \quad \eta_{ha} = (1-\alpha) \eta_{da} - \alpha N \eta_{na} / (1-N). \\ \mathbf{Case \ 2} \end{split}$$

The solutions for the elasticities given in $(\ref{eq:solution})$ for Case 2 yield a quadratic equation in k for $\eta_{ck},$

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0, (A.9)$$

where,

$$Q_2 \equiv (\psi_3 \mu_3 - \mu_3), \quad Q_1 \equiv (1 + \psi_1 \mu_3 + \psi_3 \mu_1 - \mu_1), \quad Q_0 \equiv \psi_1 \mu_1.$$

where,

$$\psi_1 \equiv (\nu_1 - 1)\lambda_3\sigma, \quad \psi_2 \equiv (\nu_2 - 1)\lambda_3\sigma, \quad \psi_3 \equiv \nu_3\lambda_3\sigma,$$

and where,

$$\mu_1 \equiv \lambda_2 \nu_1 + \lambda_1, \quad \mu_2 \equiv \lambda_2 \nu_1 + \lambda_2, \quad \mu_3 \equiv 1 - \lambda_1 - \lambda_2 + \lambda_2 \nu_3,$$

$$\nu_1 \equiv \nu^* (1 - \alpha), \quad \nu_2 \equiv \nu^* \alpha (1/\sigma), \quad \nu_3 \equiv -\nu^* \sigma,$$

$$\nu^* \equiv \frac{(1 - N)\sigma}{(1 - \alpha)(1 - N)\sigma + N},$$

$$\lambda_1 \equiv \frac{1 + r}{1 + g}, \quad \lambda_2 \equiv \frac{(r + \delta)\alpha}{(1 + g)(1 - \alpha)}, \quad \lambda_3 \equiv \frac{(r + \delta)\alpha}{1 + r},$$

$$\psi_1 \equiv (\nu_1 - 1)\lambda_3\sigma, \quad \psi_2 \equiv (\nu_2 + 1)\lambda_3\sigma, \quad \psi_3 \equiv \nu_3\lambda_3\sigma.$$

The elasticity of consumption with respect to technology is a function of η_{ck} :

$$\eta_{ca} = \frac{-(\psi_1 \mu_2 + \psi_3 \mu_2 \eta_{ck} - \mu_2 \eta_{ck} + \psi_2 \phi)}{\psi_1 \mu_3 + \psi_3 \mu_3 \eta_{ck} - \mu_3 \eta_{ck} + \psi_3 \phi + 1 - \phi},\tag{A.10}$$

and the rest of the elasticities are defined in terms of the consumption elasticities:

$$\begin{split} \eta_{kk} &= \mu_1 + \mu_3 \eta_{ck}, \quad \eta_{ka} = \mu_2 + \mu_3 \eta_{ca}, \\ \eta_{nk} &= \nu_1 + \nu_3 \eta_{ck}, \quad \eta_{na} = \nu_2 + \nu_3 \eta_{ca}, \\ \eta_{yk} &= 1 - \alpha + \alpha \eta_{nk}, \quad \eta_{ya} = \alpha + \alpha \eta_{na}, \\ \eta_{hk} &= -N \eta_{nk} / (1 - N), \quad \eta_{ha} = \alpha - N \eta_{na} / (1 - N). \\ \mathbf{Case \ 1} \end{split}$$

Parameter definitions for Case 1 are the same as in Case 2, with the following exceptions:

$$\nu_2 \equiv \nu^* \alpha, \quad \eta_{ha} = -N\eta_{na}/(1-N).$$