# Appendix for "Elasticities of Substitution in Real Business Cycle Models with Home Production"* 

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June 7, 2000

# Appendix: Approximate Loglinear Solutions for the paper Elasticities of Substitution in Real Business Cycle Models with Home Production 

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This appendix provides illustrative solutions to the loglinearized model for Cases 1 through 4. Cases 5 and 6 are straightforward generalizations of Cases 3 and 4 . We use the method of Campbell (1994). Here we provide only the solutions for the elasticities, and refer the reader to Campbell (1994) for details about the procedure. In each case, the model's equations are made linear in logs by approximating them with first order Taylor expansions around steady state values. We start with the most general case and proceed backwards. Throughout this appendix we use the notation $\sigma=1 / \gamma$.

## Case 4

Combining (??) and (??) we get an accumulation equation for $F_{t} \equiv K_{t}+D_{t}$ :

$$
\begin{equation*}
F_{t+1}=(1-\delta) F_{t}+Y_{t}-C_{t} . \tag{A.1}
\end{equation*}
$$

Taking logs of both sides and linearizing the right hand side yields an equation for $f_{t+1}$ :

$$
\begin{equation*}
f_{t+1}=\lambda_{1} k_{t}+\lambda_{2}\left(a_{t}+n_{t}\right)+\lambda_{3} c_{t} \lambda_{4} f_{t} \tag{A.2}
\end{equation*}
$$

where,

$$
\lambda_{1} \equiv \frac{(r+\delta) N}{1+g}, \quad \lambda_{2} \equiv \frac{(r+\delta) N \alpha}{(1+g)(1-\alpha)}, \quad \lambda_{3} \equiv \frac{(\delta+g)}{1+g}-\frac{(r+\delta) N}{(1+g)(1-\alpha)}, \quad \lambda_{4} \equiv 1-\frac{\delta+g}{1+g} .
$$

log-linearizing the work-wage first-order condition (??) yields an equation for log hours:

$$
\begin{equation*}
n_{t}=\nu_{1} k_{t}+\nu_{2} d_{t}+\nu_{3} a_{t}+\nu_{4} c_{t}, \tag{A.3}
\end{equation*}
$$

where,

$$
\nu_{1} \equiv \nu^{*}(1-\alpha), \quad \nu_{2} \equiv \nu^{*}(1 / \sigma-1)(1-\alpha), \quad \nu_{3} \equiv \nu^{*} \alpha / \sigma, \quad \nu_{4} \equiv-\nu^{*} / \sigma
$$

and where,

$$
\nu^{*} \equiv(\nu(1-N) \sigma) /((1-N) \sigma+\nu \alpha N), \quad \nu \equiv(1-N) /(1-\alpha) .
$$

Equation (??) is loglinearized assuming that $R_{t+1}$ and $C_{t+1}$ are jointly lognormal and homoskedastic to obtain:

$$
\begin{equation*}
E_{t} \Delta c_{t}=E_{t}\left[\xi_{1} k_{t+1}+\xi_{2} d_{t+1}+\xi_{3} a_{t+1}+\xi_{4} c_{t+1}\right] \tag{A.4}
\end{equation*}
$$

where,

$$
\begin{array}{rlrl}
\xi_{1} & \equiv\left(\sigma \alpha(r+\delta)\left(\nu_{1}-1\right)\right) /(1+r), & & \xi_{2} \equiv\left(\sigma \alpha(r+\delta) \nu_{2}\right) /(1+r) \\
\xi_{3} & \equiv\left(\sigma \alpha(r+\delta)\left(\nu_{3}+1\right)\right) /(1+r), & \xi_{4} \equiv\left(\sigma \alpha(r+\delta) \nu_{4}\right) /(1+r)
\end{array}
$$

We assume that individuals can reallocate capital between the home and market sectors within the period. This allows us to equate the gross marginal products of each type of capital in (??) and (??) yielding an equation for $k_{t}$ and $d_{t}$ in terms of $f_{t}, a_{t}$, and $c_{t}$ :

$$
\begin{align*}
& k_{t}=\pi_{1} f_{t}+\pi_{2} a_{t}+\pi_{3} c_{t}  \tag{A.5}\\
& d_{t}=\chi_{1} f_{t}+\chi_{2} a_{t}+\chi_{3} c_{t} \tag{A.6}
\end{align*}
$$

where,

$$
\begin{gathered}
\chi_{1} \equiv \omega_{1}^{*} \pi_{1}, \quad \chi_{2} \equiv \omega_{1}^{*} \pi_{2}+\omega_{2}^{*}, \quad \chi_{3} \equiv \omega_{1}^{*} \pi_{3}+\omega_{3}^{*} \\
\omega_{1}^{*} \equiv\left(\omega_{1} \nu_{1}+\omega_{3}\right) /\left(1-\omega_{1} \nu_{2}\right), \quad \omega_{2}^{*} \equiv\left(\omega_{1} \nu_{3}+\omega_{2}\right) /\left(1-\omega_{1} \nu_{2}\right) \\
\omega_{3}^{*} \equiv\left(\omega_{1} \nu_{4}+\omega_{4}\right) /\left(1-\omega_{1} \nu_{2}\right)
\end{gathered}
$$

and where,

$$
\begin{gathered}
\omega_{1} \equiv-(N /(1-N) \alpha(1-1 / \sigma) \sigma /((1-\alpha)+\alpha \sigma)-\alpha \sigma /((1-\alpha)+\alpha \sigma), \\
\omega_{2} \equiv-\alpha /((1-\alpha)+\alpha \sigma), \quad \omega_{3} \equiv \alpha \sigma /((1-\alpha)+\alpha \sigma), \quad \omega_{4} \equiv 1 /((1-\alpha)+\alpha \sigma), \\
\pi_{1} \equiv(1 / N) /\left(1+(1-N) \omega_{1}^{*} / N\right), \quad \pi_{2} \equiv-(1-N) \omega_{2}^{*} /\left(N+(1-N) \omega_{1}^{*}\right), \\
\pi_{3} \equiv-\omega_{3}^{*}(1-N) /\left(N+(1-N) \omega_{1}^{*}\right) .
\end{gathered}
$$

The solution proceeds by the method of undetermined coefficients, by making an initial guess that the loglinear solution will be of the form specified in (??).
$\eta_{c f}$ solves the quadratic equation:

$$
\begin{equation*}
Q_{2} \eta_{c f}^{2}+Q_{1} \eta_{c f}+Q_{0}=0 \tag{A.7}
\end{equation*}
$$

where,

$$
Q_{2} \equiv\left(\psi_{3} \mu_{3}-\mu_{3}\right), \quad Q_{1} \equiv\left(1+\psi_{1} \mu_{3}+\psi_{3} \mu_{1}-\mu_{1}\right), \quad Q_{0} \equiv \psi_{1} \mu_{1}
$$

where,

$$
\psi_{1} \equiv \xi_{1} \pi_{1}+\xi_{2} \chi_{1}, \quad \psi_{2} \equiv \xi_{1} \pi_{2}+\xi_{2} \chi_{2}+\xi_{3}, \quad \psi_{3} \equiv \xi_{1} \pi_{3}+\xi_{2} \chi_{3}+\xi_{4}
$$

and where,

$$
\mu_{1} \equiv \lambda_{1}^{*} \pi_{1}+\lambda_{3}^{*} \chi_{1}+\lambda_{5}^{*}, \quad \mu_{2} \equiv \lambda_{1}^{*} \pi_{2}+\lambda_{3}^{*} \chi_{2}+\lambda_{2}^{*}, \quad \mu_{3} \equiv \lambda_{1}^{*} \pi_{3}+\lambda_{3}^{*} \chi_{3}+\lambda_{4}^{*}
$$

where,

$$
\lambda_{1}^{*} \equiv \lambda_{1}+\lambda_{2} \nu_{1}, \quad \lambda_{2}^{*} \equiv \lambda_{2}+\lambda_{2} \nu_{3}, \quad \lambda_{3}^{*} \equiv \lambda_{2} \nu_{2}, \quad \lambda_{4}^{*} \equiv \lambda_{3}+\lambda_{2} \nu_{4}, \quad \lambda_{5}^{*} \equiv \lambda_{4} .
$$

$\eta_{c a}$ is given by

$$
\begin{equation*}
\eta_{c a}=\frac{-\left(\psi_{1} \mu_{2}+\psi_{3} \mu_{2} \eta_{c f}-\mu_{2} \eta_{c f}+\psi_{2} \phi\right)}{\psi_{1} \mu_{3}+\psi_{3} \mu_{3} \eta_{c f}-\mu_{3} \eta_{c f}+\psi_{3} \phi+1-\phi} \tag{A.8}
\end{equation*}
$$

Elasticities of total capital with respect to last period's total capital and the log technology shock are then found as

$$
\eta_{f f}=\mu_{1}+\mu_{3} \eta_{c f}, \quad \eta_{f a}=\mu_{2}+\mu_{3} \eta_{c a}
$$

All other elasticities are defined in terms of the elasticities above:

$$
\begin{gathered}
\eta_{k f}=\pi_{1}+\pi_{3} \eta_{c f}, \quad \eta_{k a}=\pi_{2}+\pi_{3} \eta_{c a}, \quad \eta_{d f}=\chi_{1}+\chi_{3} \eta_{c f}, \quad \eta_{d a}=\chi_{2}+\chi_{3} \eta_{c a}, \\
\eta_{n f}=\nu_{1} \eta_{k f}+\nu_{2} \eta_{d f}+\nu_{4} \eta_{c f}, \quad \eta_{n a}=\nu_{1} \eta_{k a}+\nu_{2} \eta_{d a}+\nu_{3}+\nu_{4} \eta_{c a}, \\
\quad \eta_{y f}=\alpha \eta_{n f}+(1-\alpha) \eta_{k f}, \quad \eta_{y a}=\alpha\left(1+\eta_{n a}\right)+(1-\alpha) \eta_{k a} \\
\eta_{h f}=(1-\alpha) \eta_{d f}-\alpha N \eta_{n f} /(1-N), \quad \eta_{h a}=(1-\alpha) \eta_{d a}+\alpha-\alpha N \eta_{n a} /(1-N) .
\end{gathered}
$$

## Case 3

Parameter definitions for Case 3 are the same as in Case 4, with the following exceptions:

$$
\omega_{2} \equiv-\alpha \sigma /((1-\alpha)+\alpha \sigma), \quad \nu_{3} \equiv \nu^{*} \alpha, \quad \eta_{h a}=(1-\alpha) \eta_{d a}-\alpha N \eta_{n a} /(1-N)
$$

## Case 2

The solutions for the elasticities given in (??) for Case 2 yield a quadratic equation in $k$ for $\eta_{c k}$,

$$
\begin{equation*}
Q_{2} \eta_{c k}^{2}+Q_{1} \eta_{c k}+Q_{0}=0 \tag{A.9}
\end{equation*}
$$

where,

$$
Q_{2} \equiv\left(\psi_{3} \mu_{3}-\mu_{3}\right), \quad Q_{1} \equiv\left(1+\psi_{1} \mu_{3}+\psi_{3} \mu_{1}-\mu_{1}\right), \quad Q_{0} \equiv \psi_{1} \mu_{1}
$$

where,

$$
\psi_{1} \equiv\left(\nu_{1}-1\right) \lambda_{3} \sigma, \quad \psi_{2} \equiv\left(\nu_{2}-1\right) \lambda_{3} \sigma, \quad \psi_{3} \equiv \nu_{3} \lambda_{3} \sigma,
$$

and where,

$$
\begin{gathered}
\mu_{1} \equiv \lambda_{2} \nu_{1}+\lambda_{1}, \quad \mu_{2} \equiv \lambda_{2} \nu_{1}+\lambda_{2}, \quad \mu_{3} \equiv 1-\lambda_{1}-\lambda_{2}+\lambda_{2} \nu_{3}, \\
\nu_{1} \equiv \nu^{*}(1-\alpha), \quad \nu_{2} \equiv \nu^{*} \alpha(1 / \sigma), \quad \nu_{3} \equiv-\nu^{*} \sigma \\
\nu^{*} \equiv \frac{(1-N) \sigma}{(1-\alpha)(1-N) \sigma+N}, \\
\lambda_{1} \equiv \frac{1+r}{1+g}, \quad \lambda_{2} \equiv \frac{(r+\delta) \alpha}{(1+g)(1-\alpha)}, \quad \lambda_{3} \equiv \frac{(r+\delta) \alpha}{1+r} \\
\psi_{1} \equiv\left(\nu_{1}-1\right) \lambda_{3} \sigma, \quad \psi_{2} \equiv\left(\nu_{2}+1\right) \lambda_{3} \sigma, \quad \psi_{3} \equiv \nu_{3} \lambda_{3} \sigma .
\end{gathered}
$$

The elasticity of consumption with respect to technology is a function of $\eta_{c k}$ :

$$
\begin{equation*}
\eta_{c a}=\frac{-\left(\psi_{1} \mu_{2}+\psi_{3} \mu_{2} \eta_{c k}-\mu_{2} \eta_{c k}+\psi_{2} \phi\right)}{\psi_{1} \mu_{3}+\psi_{3} \mu_{3} \eta_{c k}-\mu_{3} \eta_{c k}+\psi_{3} \phi+1-\phi} \tag{A.10}
\end{equation*}
$$

and the rest of the elasticities are defined in terms of the consumption elasticities:

$$
\begin{gathered}
\eta_{k k}=\mu_{1}+\mu_{3} \eta_{c k}, \quad \eta_{k a}=\mu_{2}+\mu_{3} \eta_{c a}, \\
\eta_{n k}=\nu_{1}+\nu_{3} \eta_{c k}, \quad \eta_{n a}=\nu_{2}+\nu_{3} \eta_{c a}, \\
\eta_{y k}=1-\alpha+\alpha \eta_{n k}, \quad \eta_{y a}=\alpha+\alpha \eta_{n a} \\
\eta_{h k}=-N \eta_{n k} /(1-N), \quad \eta_{h a}=\alpha-N \eta_{n a} /(1-N) .
\end{gathered}
$$

## Case 1

Parameter definitions for Case 1 are the same as in Case 2, with the following exceptions:

$$
\nu_{2} \equiv \nu^{*} \alpha, \quad \eta_{h a}=-N \eta_{n a} /(1-N)
$$

